Math 4200
Monday September 21
2.1-2.2
On Friday we had defined integrals of complex-valued functions $f(t)$ from real number domains, i.e. $f: [a, b] \rightarrow \mathbb{C}$. And we computed one such integral. We'll pick that discussion up today, and use those elementary integrals to define the more specialized *contour integrals* that are a key construction in complex analysis, and which are actually complex versions of *line integrals*

$$\int_{\gamma} P\, dx + Q\, dy$$

that you've worked with in multivariable calculus classes.

Announcements:

(green is from Friday)

2.1 Integration of complex-valued functions of a real variable t, just as in Calc 1.
Introduction to *contour integrals* - analogous to *line integrals* from multivariable Calculus.

A1 Def: For $f:[a,b]\subseteq\mathbb{R}\to\mathbb{C}$ continuous, $f(t)=u(t)+i\,v(t)$, with $u=\mathrm{Re}(f)$, $v=\mathrm{Im}(f)$

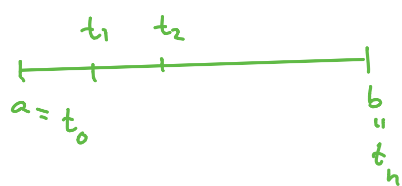
- $$\int_a^b f(t)\,dt=\int_a^b u(t)+i\,v(t)\,dt:=\int_a^b u(t)\,dt+i\int_a^b v(t)\,dt.$$

It is useful for estimates to note that since u, v are continuous on $[a,b]$ they are uniformly continuous - and you proved in Math 3210 that in this case definite integrals are limits of Riemann sums for partionings P of $[a,b]$, as the "norm" of the partition approaches zero: For

$$P:=\quad a=t_0<t_1<\ldots<t_n=b$$

$j=0,\ldots n-1$ • $t_j\le t_j^*\le t_{j+t}$, $\quad\Delta t_j=t_{j+1}-t_j$

$$\|P\|:=\max\Delta t_j\,,$$



3210 $$\int_a^b u(t)\,dt=\lim_{\|P\|\to0}\sum_j u\left(t_j^*\right)\Delta t_j,\qquad\int_a^b v(t)\,dt=\lim_{\|P\|\to0}\sum_j v\left(t_j^*\right)\Delta t_j$$

so also
A2 Def :

- $$\int_a^b f(t)\,dt=\lim_{\|P\|\to0}\sum_j u\left(t_j^*\right)\Delta t_j+i\lim_{\|P\|\to0}\sum_j v\left(t_j^*\right)\Delta t_j=\lim_{\|P\|\to0}\sum_j f\left(t_j^*\right)\Delta t_j.$$

Example 1: Use Calc 1 FTC to compute

$$\int_0^{\frac{\pi}{2}}\underbrace{-2\sin(t)\cos(t)}_{-\sin 2t}+i\underbrace{\left(\cos^2(t)-\sin^2(t)\right)}_{\cos 2t}\,dt.$$

$$=\int_0^{\pi/2}-\sin 2t\,dt+i\int_0^{\pi/2}\cos 2t\,dt$$

$$\left.\frac{\cos 2t}{2}\right]_0^{\pi/2}+i\left.\frac{\sin 2t}{2}\right]_0^{\pi/2}$$

$$=\frac{1}{2}(\cos\pi-\cos 0)+\frac{i}{2}(\sin\pi-\sin 0)$$

$$=-1+0=-1\quad !!$$

What we did in the previous example works in general:

A3 Fundamental Theorem of Calculus for $f\colon [a, b] \to \mathbb{C}$: Let $u, v\colon [a, b] \to \mathbb{R}$ continuous, $f(t) = u(t) + i\,v(t)$, $F(t)$ such that $F'(t) = f(t)$. Then

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a).$$

Use the triangle inequality on Riemann sums to prove the important integral estimate which bounds the modulus of definite integrals in terms of the integrals of their modulus:

A4 <u>Theorem</u>

$$\left|\int_a^b f(t)\,\mathrm{d}t\right| \le \int_a^b |f(t)|\,\mathrm{d}t.$$

B1 Def Let $A \subseteq \mathbb{C}$ open, $f : A \to \mathbb{C}$ continuous, $\gamma : [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ a C^1 curve. Then the *complex line integral* or *contour integral*

$$\int_\gamma f(\boldsymbol{z})\ \mathrm{d}\boldsymbol{z} := \int_a^b f(\gamma(t))\gamma'(t)\ \mathrm{d}t$$

where we use the definition *A1* on the previous page to compute the integral on the right. Note, we have substituted $\boldsymbol{z} = \gamma(t)$ and used the differential substitution, $d\boldsymbol{z} = \gamma'(t)dt$ into the integrand.

B2 In the case that $\gamma'(t) \neq 0$ for any t it follows from the continuity of $|\gamma'(t)|$ that $|\gamma'(t)| \geq \delta > 0$ on $[a, b]$. And in this case the complex line integral above can be realized as a limit which explains the geometry of what's going on:

$$\int_\gamma f(z)\ \mathrm{d}z = \lim_{\max\{|\Delta z_j|\} \to 0} \sum_{j=0}^{n-1} f(z_j)\Delta z_j \ ,$$

$$P := \ a = t_0 < t_1 < \ldots < t_n = b;$$

$$\Delta t_j = t_{j+1} - t_j \qquad \|P\| := \max \Delta t_j$$

$$z_j = \gamma(t_j), \quad \Delta z_j = z_{j+1} - z_j = \gamma(t_{j+1}) - \gamma(t_j).$$

The reason this is true is that by the 3210 or 3220 affine approximation formula for the C^1 curve γ,

$$\gamma(t_{j+1}) - \gamma(t_j) = \gamma'(t_j)\Delta t_j \ + \varepsilon(t_j)\Delta t_j$$

where one can show that the $|\varepsilon(t)| \to 0$ uniformly as $\|P\| \to 0$ because γ is continuously differentiable. Also, because $M \geq |\gamma'(t)| \geq \delta$ the condition that $\max\{|\Delta z_j|\} \to 0$ in $\mathbb{C}$is equivalent to the $\|P\| \to 0$ in $[a, b]$, also because of the approximation formula. So,

$$\lim_{\max\{|\Delta z_j|\} \to 0} \sum_{j=0}^{n-1} f(z_j)\Delta z_j \ = \lim_{\|P\| \to 0} \sum_j f(\gamma(t_j))\,(\gamma(t_{j+1}) - \gamma(t_j))$$

$$= \lim_{\|P\| \to 0} \sum_j f(\gamma(t_j))\,(\gamma'(t_j)\Delta t_j + \varepsilon(t_j)\Delta t_j)$$

$$= \lim_{\|P\| \to 0} \sum_j f(\gamma(t_j))\,\gamma'(t_j)\Delta t_j$$

$$= \int_a^b f(\gamma(t))\gamma'(t)\ \mathrm{d}t\ .$$

Example 2: Let $\gamma(t) = e^{it}$, $0 \leq t \leq \frac{\pi}{2}$, $f(z) = z$. Compute

$$\int_{\gamma} f(z)\, dz.$$

Sketch. Do you think you would get the same answer if you followed the same quarter circle in the same direction, but with a different parameterization? What if you reversed direction? Could you explain why?

B3 Theorem FTC for contour integrals Let $A \subseteq \mathbb{C}$ open, $f: A \to \mathbb{C}$ continuous, $\gamma: [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ a C^1 curve. If f has an analytic antiderivative in A, i.e. $F' = f$, then complex line integrals only depend on the endpoints of the curve γ, via the formula

$$\int_\gamma f(\mathbf{z})\, \mathrm{d}\mathbf{z} := F(\gamma(b)) - F(\gamma(a))$$

proof:

Example 3: Redo Example 2 using the FTC for contour integrals:

$\gamma(t) = \mathrm{e}^{i\,t},\ 0 \le t \le \frac{\pi}{2},\ f(\mathbf{z}) = \mathbf{z},$

$$\int_\gamma \mathbf{z}\, \mathrm{d}\mathbf{z} =$$

B4 Integral estimate: Let $A \subseteq \mathbb{C}$ open, $f: A \to \mathbb{C}$ continuous, $\gamma: [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ a C^1 curve. Then

$$\left| \int_\gamma f(z)\, \mathrm{d}z \right| = \left| \int_a^b f(\gamma(t))\gamma'(t)\, \mathrm{d}t \right|$$

$$\leq \int_a^b \left| f(\gamma(t))\gamma'(t) \right| \mathrm{d}t \qquad (A4)$$

$$= \int_a^b \left| f(\gamma(t)) \right| \left| \gamma'(t) \right| dt$$

Def Let $A \subseteq \mathbb{C}$ open, $f: A \to \mathbb{C}$ continuous, $\gamma: [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ a C^1 curve. Then

$$\int_\gamma |f(z)|\, |\mathrm{d}z| := \int_a^b \left| f(\gamma(t)) \right| \left| \gamma'(t) \right| dt$$

Using the definition, we see that the shorthand for the integral estimate in *B3* is

$$\left| \int_\gamma f(z)\, \mathrm{d}z \right| \leq \int_\gamma |f(z)|\, |\mathrm{d}z|.$$

Note that $|dz| = |\gamma'(t)|\, dt$ is the element of arclength.

Example 4: In the running example we showed that for $\gamma(t) = e^{it},\ 0 \leq t \leq \frac{\pi}{2}$,

$$\int_\gamma z\, \mathrm{d}z = -1.$$

Compute

$$\int_\gamma |z|\, |\mathrm{d}z|$$

and verify the integral estimate *B4*.

Example 5. Consider a circle of positive radius a centered at any point $z_0 \in \mathbb{C}$. Find an appropriate parameterization which traverses this circle once in the counterclockwise direction and verify one of the most-used contour integral equalities in complex analysis:

$$\int_{|z - z_0| = a} \frac{1}{z - z_0}\, dz = 2\pi i.$$

5b) In an effort to tie this computation in to the FTC for contour integrals, could you compute this integral in that way? (The answer is yes, if you're careful!)

The connection between contour integrals and Calc 3 line integrals:

Let $A \subseteq \mathbb{C}$ open, $f: A \rightarrow \mathbb{C}$ continuous, $\gamma: [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{C}$ a C^1 curve. write

$$\gamma(t) = x(t) + i\,y(t),$$
$$f(z) = u(x, y) + i\,v(x, y).$$

Then

$$\int_\gamma f(z)\ \mathrm{d}z = \int_a^b f\big(\gamma(t)\big)\gamma'(t)\ \mathrm{d}t$$

$$= \int_a^b \big(u(x(t), y(t)) + i\,v(x(t), y(t))\big)\big(x'(t) + i\,y'(t)\big)\ \mathrm{d}t$$

$$= \int_a^b u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t)dt$$

$$+\ i\int_a^b v(x(t), y(t))x'(t) + u(x(t), y(t))y'(t)dt$$

$$= \int_\gamma u\,dx - v\,dy + i\int_\gamma v\,dx + u\,dy.$$

On Wednesday we'll combine this Calc 3 line integral way of writing contour integrals with Calc 3 Green's Theorem, for some interesting section 2.2 results.